HOMEWORK 2 SOLUTIONS

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Problem 3.2

Suppose a divides b and b divides c for some integers a, b, and c. Since a divides b, there exists an integer k such that $b = ka$. Since b divides c, there exists an integer m such that $c = mb$. Putting this together, we can write $c = mb = m(ka) = (mk)a$. In other words, we've expressed c as an integer times a. Therefore a divides c.

Problem 3.3

Let n be an even integer. Then 2 divides n, so there exists an integer k such that $n = 2k$. Therefore $n^2 = (2k)^2 = 2(2k^2)$, so we conclude that 2 divides n^2. Therefore n^2 is even.

Problem 3.5

Let $a > 0$ and suppose $b \geq c$. Recall that $b \geq c$ means $b > c$ or $b = c$. We'll consider each case separately. First suppose $b > c$. Since $a > 0$, the multiplication law implies $ab > ac$. Now suppose $b = c$. Multiplying both sides by a gives $ab = ac$. From these two cases we conclude $ab > ac$ or $ab = ac$, which is the same as saying $ab \geq ac$ as desired.

Problem 3.8

(i) Let a and b be real numbers such that $0 \leq a < b$. Since $a \geq 0$ there are two cases to consider: $a = 0$ and $a > 0$. First suppose $a = 0$. Then $0 < b$. By the multiplication law, we can multiply both sides by b to conclude that $0b < b^2$. Thus, $a^2 = 0^2 = 0 = 0b < b^2$, so by transitivity we have $a^2 < b^2$.

Now suppose $a > 0$. By the multiplication law, $a < b$ implies $a^2 < ab$. Since $0 < a < b$ the transitive law implies $b > 0$. By the multiplication law, $a < b$ implies $ab < b^2$. Using the transitive law again, $a^2 < ab$ and $ab < b^2$ implies that $a^2 < b^2$ as desired.

(ii) Let a and b be real numbers such that $|a| < |b|$. Since $0 \leq |a| < |b|$, part (i) implies that $|a|^2 < |b|^2$. By problem 1.5 of Homework 1, $|a|^2 = a^2$ for every real number a. Thus, $|a|^2 < |b|^2$ is equivalent to $a^2 < b^2$.

(iii) Let a and b be real numbers such that $|a| = |b|$. Then $|a|^2 = |b|^2$. By problem 1.5 of Homework 1, $|a|^2 = a^2$ and $|b|^2 = b^2$, so we conclude that $a^2 = b^2$.

(iv) Let a and b be real numbers such that $|a| \leq |b|$. Then either $|a| = |b|$ or $|a| < |b|$. If $|a| = |b|$ then by part (iii) we may conclude $a^2 = b^2$. If $|a| < |b|$ then by part (ii) we may conclude $a^2 < b^2$. Since $a^2 = b^2$ or $a^2 < b^2$, we have that $a^2 \leq b^2$ as desired.

Problem 4.1

Suppose toward a contradiction that there do exist integers m and n such that $14m + 21n = 100$. Then we have $7(2m+3n) = 14m+21n = 100$. Since $2m+3n$ is an integer, 7 divides 100. By problem 2.6 of Homework 1, 7 does not divide 100. This is a contradiction. Therefore there do not exist integers m and n such that $14m + 21n = 100$.

Problem 4.3

Remember: the contrapositive of the statement $P \implies Q$ is $\neg Q \implies \neg P$. These statements are logically equivalent, so to prove one it is enough to prove the other. We'll prove "n^2 is odd $\implies$ n is odd" by contraposition. The contrapositive of this statement is "n is not odd $\implies$ n^2 is not odd", or equivalently, if n is even then n^2 is even. Fortunately, we already proved this in problem 3.3 of this homework! I'll give the proof again anyway, for completeness:

Let n be an even integer. Then 2 divides n, so there exists an integer k such that $n = 2k$. Therefore $n^2 = (2k)^2 = 2(2k^2)$, so we conclude that 2 divides n^2. Therefore n^2 is even. Since we've shown that the contrapositive of the original statement is true, we may conclude that the original statement is true as well.

Problem 4.4

Let a be a real number such that $a^2 \geq 7a$. To see that it follows that $a \leq 0$ or $a \geq 7$, suppose it is not the case that $a \leq 0$. In other words, that $a > 0$. Since $a^2 \geq 7a$ we have $a^2 = 7a$ or $a^2 > 7a$.

Case 1: $a^2 = 7a$. Since $a \neq 0$ we may divide both sides by a to conclude that $a = 7$.

Case 2: $a^2 > 7a$. Note that since $a > 0$, we have $1/a > 0$ as well. By the multiplication law, we can multiply both sides of the inequality by $1/a$ to conclude that $a > 7$.
Thus, $a = 7$ or $a > 7$, so we conclude that $a \geq 7$.

Problem 4.7

We'll first prove the following claim, which will be useful:

Claim: For all real numbers a and b, $a^2 \leq b^2$ implies $|a| \leq |b|$.

Proof: Suppose toward a contradiction that $|a|^2 \leq b^2$ and $|a| > |b|$. By problem 3.8 (ii), this implies $a^2 > b^2$. This contradicts the fact that $a^2 \leq b^2$. Therefore $a^2 \leq b^2$ implies $|a| \leq |b|$.

Now back to the original problem: By the previous claim, it will suffice to prove that $|a + b|^2 \leq (|a| + |b|)^2$. To this end, we have

$$\begin{aligned} |a+b|^2 &= (a+b)^2 \text{ by problem 1.5} \\ &= a^2 + 2ab + b^2 \end{aligned}$$

and

$$\begin{aligned}(|a| + |b|)^2 &= |a|^2 + 2|a||b| + |b|^2 \\ &= a^2 + 2|a||b| + b^2 \text{ by problem 1.5}\end{aligned}$$

Thus, we need to show that $a^2 + 2ab + b^2 \leq a^2 + 2|a||b| + b^2$. Subtracting $a^2 + b^2$ from both sides, this is equivalent to $2ab \leq 2|a||b|$. Next we'll show $|a||b| = |ab|$. To see this, observe that

$$(|a||b|)^2 = |a|^2|b|^2 = a^2b^2 = (ab)^2 = |ab|^2.$$

Taking a square root on both sides and noting that all terms are nonnegative gives $|a||b| = |ab|$. Since $ab \leq |ab|$, we have $2ab \leq 2|ab| = 2|a||b|$, which proves the original result.

Now we'll determine necessary and sufficient conditions for equality. The only part of our proof which involved an inequality rather than an equality was in showing $ab \leq |ab|$. Equality holds here precisely when $ab \geq 0$. In other words, equality holds if and only if a and b have the same sign.